

# ON THE STRUCTURE OF THOM POLYNOMIALS OF SINGULARITIES

L. M. FEHÉR AND R. RIMÁNYI

**ABSTRACT.** Thom polynomials of singularities express the cohomology classes dual to singularity submanifolds. A stabilization property of Thom polynomials is known classically, namely that trivial unfolding does not change the Thom polynomial. In this paper we show that this is a special case of a product rule. The product rule enables us to calculate the Thom polynomials of singularities if we know the Thom polynomial of the product singularity. As a special case of the product rule we define a formal power series (Thom series,  $\text{Ts}_Q$ ) associated with a commutative, complex, finite dimensional local algebra  $Q$ , such that the Thom polynomial of *every* singularity with local algebra  $Q$  can be recovered from  $\text{Ts}_Q$ .

## 1. INTRODUCTION

For a holomorphic map  $f : N \rightarrow M$  between complex manifolds, and a singularity  $\eta$ , one can consider the points in  $N$  where the map has singularity  $\eta$ . In [Tho56] René Thom noticed that the cohomology class represented by this set can be calculated by a universal polynomial depending only on the singularity, if we substitute the characteristic classes of the map. In modern language the singularity  $\eta$  is a subset of the vector space of all germs  $(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^m, 0)$ , and its Thom polynomial is its Poincaré dual in equivariant cohomology (see the group action below). Different methods to compute certain concrete Thom polynomials (resolution, interpolation, localization, Gröbner basis, etc) have been found recently, and the enumerative and combinatorial application of the computed Thom polynomials remains a hot area in geometry and algebraic combinatorics.

In this paper we study the interior structure of Thom polynomials of singularities. We carry out a program, similar in spirit to the ‘Thom-Sebastiani’ program of local singularity theory, by studying Thom polynomials of product singularities. Our main example uncovers a so far hidden strong constraint on Thom polynomials.

Using this new property we reduce the knowledge of Thom polynomials of ‘contact’ singularities with a given local algebra to the knowledge of a formal power series, that we named Thom series. This is the content of Theorem 4.1 below. Namely, under the conditions listed there, a finite dimensional commutative local  $\mathbf{C}$ -algebra  $Q$  gives rise to a formal power series  $\text{Ts}_Q$  in the variables  $d_0, d_{\pm 1}, d_{\pm 2}, \dots$ , such that the Thom polynomial of any singularity  $\eta : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^{n+k}, 0)$  with local algebra  $Q$  can be obtained from  $\text{Ts}_Q$  by substituting  $d_i = c_{i+k+1}$ . For example, for  $Q = \mathbf{C}[x]/(x^3)$  we have  $\text{Tp}_Q = d_0^2 + d_{-1}d_1 + 2d_{-2}d_2 + 4d_{-3}d_3 + \dots$ , hence the Thom polynomial of a singularity  $\eta : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^n, 0)$  with local algebra  $Q$  is  $c_1^2 + c_2$ ; the Thom polynomial of a singularity  $\eta : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^{n+1}, 0)$  with local algebra  $Q$  is  $c_2^2 + c_1c_3 + 2c_4$ , etc.

The authors thank A. Buch, M. Kazarian, B. Kőmüves, and A. Némethi for valuable discussions.

---

Supported by OTKA T046365MAT, and NSF grant DMS-0405723 (2nd author)  
 Keywords: Thom polynomial, equivariant maps  
 AMS Subject classification 32S20.

## 2. THOM POLYNOMIALS AND RELATIONS BETWEEN THEM

In this paper we use cohomology with rational coefficients.

Let the Lie group  $G$  act on the complex vector space  $V$  with a  $G$ -invariant irreducible complex subvariety  $\eta \subset V$  of real codimension  $l$ . The Poincaré dual of  $\eta$  in  $G$ -equivariant cohomology is called the Thom polynomial of  $\eta$  and is denoted by  $\text{Tp}(\eta) \in H_G^*(V) = H^*(BG)$ . More generally, the kernel of the restriction ring homomorphism  $H_G^*(V) \rightarrow H_G^*(V \setminus \eta)$  is called the avoiding ideal of  $\eta$  and will be denoted by  $\mathcal{A}_\eta$ . The avoiding ideal has the following remarkable property

$$\mathcal{A}_\eta \cap H^i(BG) = \begin{cases} 0 & \text{if } i < l \\ \mathbf{Q}, \text{ spanned by } \text{Tp}(\eta) & \text{if } i = l. \end{cases}$$

For more general definitions, discussions, and the proof of this property see [FR04]. (The avoiding ideal is not necessarily principal; it just shares the property of homogeneous principal ideals that the lowest degree piece is 1-dimensional.)

Consider now the Lie groups  $G$  and  $G^\sharp = G \times H$  acting on the vector spaces  $V$  and  $V^\sharp$  respectively. Let  $j : V \rightarrow V^\sharp$  be a continuous map for which  $(g, h) \cdot j(v) = j(g \cdot v)$  for  $g \in G, h \in H, v \in V$ . This is equivalent to requiring that  $j$  is a  $G^\sharp$ -equivariant map, where the  $H$ -action on  $V$  is defined to be trivial. Let  $\eta \subset V$  and  $\eta^\sharp \subset V^\sharp$  be invariant subvarieties with  $j^{-1}(\eta^\sharp) = \eta$ , as in the diagram

$$\begin{array}{ccc} G & < & G^\sharp = G \times H \\ \curvearrowright & & \curvearrowright \\ V & \xrightarrow{j} & V^\sharp \\ \cup & & \cup \\ \eta & \longrightarrow & \eta^\sharp. \end{array}$$

**Theorem 2.1.** *Let  $p = \sum x_i \otimes y_i \in \mathcal{A}_{\eta^\sharp} \triangleleft H^*(BG^\sharp) = H^*(BG) \otimes H^*(BH)$ , where  $y_i$  is an additive basis of  $H^*(BH)$ . Then  $x_i \in \mathcal{A}_\eta \triangleleft H^*(BG)$  for every  $i$ .*

*Proof.* The rings  $H_{G^\sharp}^*(V)$  and  $H_{G^\sharp}^*(V^\sharp)$  are isomorphic to  $H^*(BG \times BH)$  (since  $V$  and  $V^\sharp$  are contractible) and  $j^*$  is an isomorphism between them. The theorem follows from the commutative diagram

$$\begin{array}{ccccc} H_{G^\sharp}^*(V) & \xleftarrow{j^*} & H_{G^\sharp}^*(V^\sharp) & \ni p = \sum x_i \otimes y_i \\ \downarrow y_i \mapsto y_i & & \downarrow & & \\ H_G^*(V \setminus \eta) \langle y_i \rangle & = & H_{G^\sharp}^*(V \setminus \eta) & \xleftarrow{\quad} & H_{G^\sharp}^*(V^\sharp \setminus \eta^\sharp), \end{array}$$

where vertical arrows are restriction homomorphisms. The equality  $H_G^*(V \setminus \eta) \langle y_i \rangle = H_{G^\sharp}^*(V \setminus \eta)$  holds, since the action of  $H$  on  $V$  is trivial.  $\square$

Now we specialize to examples in singularity theory. We will use some standard notions of singularity theory, see e.g. [AGLV98]. Namely,  $\mathcal{E}(n, m)$  will denote the infinite dimensional vector space of holomorphic germs  $(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^m, 0)$ . On this space the so-called right-left and the contact group  $\mathcal{K}(n, m)$  act. Both groups are infinite dimensional, but—in a generalized sense—they are both homotopy equivalent to  $GL(n) \times GL(m)$  [Rim02]. (Our base field is  $\mathbf{C}$  in the whole paper, i.e.  $GL(n)$  means  $GL(n, \mathbf{C})$ .)

Certain  $G = \mathcal{K}(n, m)$  invariant subsets  $\eta \subset \mathcal{E}(n, m)$  define Thom polynomials

$$\text{Tp}(\eta) \in H^*(B\mathcal{K}(n, m)) = H^*(BGL(n) \times BGL(m)) = \mathbf{Q}[a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m],$$

where  $a_i$  and  $b_i$  are the universal Chern classes of  $GL(n)$  and  $GL(m)$  respectively ( $\deg a_i = \deg b_i = 2i$ ).

It is known that the Thom polynomial  $\text{Tp}(\eta)$  of the singularity  $\eta \in \mathcal{E}(n, m)$  only depends on the quotient Chern classes, i.e. on the classes  $c_i$  defined by

$$1 + c_1 t + c_2 t^2 + \dots = \frac{1 + b_1 t + b_2 t^2 + \dots}{1 + a_1 t + a_2 t^2 + \dots},$$

where  $t$  is a formal variable. This theorem was a folklore statement in the 60's, rediscovered by many recently. The first written proof is probably in [Dam72]—we will call this statement the Thom-Damon theorem. Observe that the statement holds only for contact classes and not for right-left classes in general.

Returning to the setting of Theorem 2.1, let  $V = \mathcal{E}(n, m)$ ,  $V^\sharp = \mathcal{E}(n, m+1)$  and let  $j : V \rightarrow V^\sharp$  be defined by

$$j(f)(x_1, \dots, x_n) = (f(x_1, \dots, x_n), 0).$$

Let  $G = \mathcal{K}(n, m)$  be the contact group and  $H = U(1)$ . In the rings

$$\begin{aligned} H^*(B\mathcal{K}(n, m)) &= \mathbf{Q}[a_1, \dots, a_n, b_1, \dots, b_m], \\ H^*(B(\mathcal{K}(n, m) \times U(1))) &= \mathbf{Q}[a_1, \dots, a_n, b_1, \dots, b_m, y] \quad (\deg y = 2) \\ &= H^*(B\mathcal{K}(n, m))[y] \end{aligned}$$

we define elements  $c_i$  and  $c_i^\sharp$  by

$$\begin{aligned} 1 + c_1 t + c_2 t^2 + \dots &= \frac{\sum_{i=0}^m b_i t^i}{\sum_{i=0}^n a_i t^i}, \\ 1 + c_1^\sharp t + c_2^\sharp t^2 + \dots &= \frac{(\sum_{i=0}^m b_i t^i)(1 + yt)}{\sum_{i=0}^n a_i t^i}. \end{aligned}$$

Here and in the sequel we use the convention that  $a_0 = b_0 = 1$ ,  $a_{<0} = b_{<0} = 0$ .

Let us choose one of the following two cases for the pair  $\eta \in V$ ,  $\eta^\sharp \in V^\sharp$ .

**A:** For a finite dimensional local  $\mathbf{C}$ -algebra  $Q$  let  $\eta_0 \subset V$  and  $\eta_0^\sharp \subset V^\sharp$  be the subsets ('singularities') with associated local algebra  $Q$ . Suppose that the closures  $\eta := \overline{\eta_0}$  and  $\eta^\sharp := \overline{\eta_0^\sharp}$  define Thom polynomials.

**B:** Let  $\eta_0 = \Sigma^I(n, m) \subset V$  and  $\eta_0^\sharp = \Sigma^I(n, m+1) \subset V^\sharp$  be the collection of germs with Thom-Boardman symbol  $I$  (see [Boa67]), and let  $\eta$ ,  $\eta^\sharp$  be their closures.

In both cases denote the complex codimensions of  $\eta$  and  $\eta^\sharp$  by  $l$  and  $l^\sharp$  respectively. In both cases  $\eta$  and  $\eta^\sharp$  define Thom polynomials, and by the Thom-Damon theorem they only depend on the 'quotient' Chern classes  $c_i$  and  $c_i^\sharp$  respectively. Indeed, in case **A** this follows from the fact that two germs are contact equivalent if and only if their local algebras are isomorphic. In case **B** this follows from the fact that Thom-Boardman classes are contact invariant. We also have that  $j^{-1}(\eta^\sharp) = \eta$  so the conditions of Theorem 2.1 are satisfied. To apply Theorem 2.1 to this situation, we need the following definition and proposition.

**Definition 2.2.** Consider infinitely many variables  $x_1, x_2, \dots$  and use the convention of  $x_0 = 1$ ,  $x_{<0} = 0$ . Let  $p = x_{u_1} x_{u_2} \dots x_{u_r}$  be a monomial (with the  $u_j$ 's not necessarily different) in the variables  $x_1, x_2, \dots$  and let  $i$  be a nonnegative integer. Let  $\binom{[r]}{i}$  denote the set of  $i$ -element subsets

of  $\{1, \dots, r\}$ , and for such a subset  $I$ ,  $\delta_I$  will denote its characteristic function. We define the lowering operator  $\flat[i]$  by

$$p^{\flat[i]}(x_1, x_2, \dots) = \sum_{I \in \binom{[r]}{i}} x_{u_1 - \delta_I(1)} x_{u_2 - \delta_I(2)} \cdots x_{u_r - \delta_I(r)},$$

and we extend the definition to all polynomials linearly.

For example  $(x_1 x_2 x_5 + x_8 + x_4^2)^{\flat[2]} = x_1 x_5 + x_2 x_4 + x_1^2 x_4 + x_3^2$ .

**Proposition 2.3.** *Let  $p(x_1, x_2, \dots)$  be a degree  $l^\sharp$  polynomial for which*

$$p(c_1^\sharp, c_2^\sharp, \dots) = \sum_i p_i(c_1, c_2, \dots) y^{l^\sharp - i}.$$

*Then  $p_i = p^{\flat[i]}$ .*

*Proof.* Since  $1 + c_1^\sharp t + c_2^\sharp t^2 + \dots = (1 + c_1 t + c_2 t^2 + \dots)(1 + yt)$ , the proposition follows from the definition of  $\flat[i]$ .  $\square$

Now we are able to state the connection between the Thom polynomials  $\text{Tp}(\eta^\sharp)$  and  $\text{Tp}(\eta)$ .

**Theorem 2.4.** *Let  $\text{Tp}(\eta^\sharp) = p(c_1^\sharp, c_2^\sharp, \dots)$  and  $\text{Tp}(\eta) = q(c_1, c_2, \dots)$ . Then*

$$p^{\flat[i]} = \begin{cases} 0 & \text{if } i > l^\sharp - l \\ k_\eta \cdot q, & \text{if } i = l^\sharp - l \end{cases}$$

where  $k_\eta \in \mathbf{Q}$ .

*Proof.* The polynomial  $p^{\flat[i]}$  must belong to the avoiding ideal  $\mathcal{A}_\eta$  according to Theorem 2.1 and Proposition 2.3. According to [FR04, Sect. 2.3]—recalled at the beginning of this section—the lowest degree elements of the avoiding ideal  $\mathcal{A}(\eta)$  have degree  $l$ , and are constant multiples of the Thom polynomial  $\text{Tp}(\eta)$ .  $\square$

In fact  $k_\eta \in \mathbf{Z}$  since the cohomology of  $BGL(n)$  has no torsion so we can work here with integer coefficients.

**Corollary 2.5.** *The number of factors in any term of  $\text{Tp}(\eta^\sharp)(c_1^\sharp, c_2^\sharp, \dots)$  is at most  $l^\sharp - l$ .*

*Proof.* Let the longest terms have  $i > l^\sharp - l$  factors, and let  $T$  be the lexicographically largest term with  $i$  factors. Then  $T^{\flat[i]}$  has one term, and that term will not be among the terms of  $S^{\flat[i]}$ , for the other terms  $S$ . Hence  $\text{Tp}(\eta^\sharp)^{\flat[i]} \neq 0$ , which contradicts to Theorem 2.4.  $\square$

We can assume that all terms of  $\text{Tp}(\eta^\sharp)$  have exactly  $l^\sharp - l$  factors, since we can multiply the shorter terms by an appropriate  $c_0^\sharp$ -power. The benefit is that the  $\flat[l^\sharp - l]$  operation is particularly simple for polynomials with terms of exactly  $l^\sharp - l$  factors: it decreases all indices by 1. Hence it is worth shifting the indices in the notation: let  $d_i = c_{i+(m-n+1)}$  and  $d_i^\sharp = c_{i+(m-n+2)}^\sharp$ . Then, in the cases when  $k_\eta = 1$ , we have that the Thom polynomial of  $\eta$  (written in the  $d$ -variables) is obtained from the Thom polynomial of  $\eta^\sharp$  (written in the  $d^\sharp$ -variables) by deleting the  $\sharp$ 's. For example we have

$$\begin{aligned} \text{for } m = n + 1 \quad \text{Tp}(A_2) &= c_2^\sharp c_2^\sharp + c_1^\sharp c_3^\sharp + 2c_4^\sharp &= d_0^\sharp d_0^\sharp + d_{-1}^\sharp d_1^\sharp + 2d_{-2}^\sharp d_2^\sharp, \\ \text{for } m = n \quad \text{Tp}(A_2) &= c_1 c_1 + c_2 &= d_0 d_0 + d_{-1} d_1 + 2d_{-2} d_2 \end{aligned}$$

cf. Section 4. Observe that  $d_{-2}d_2$  is 0 in the second line, but not 0 in the first line. Hence from  $\text{Tp}(\eta^\sharp)$  we can compute  $\text{Tp}(\eta)$  but not vice versa in general.

### 3. THE CONSTANT $k_\eta$ .

In this section we study the constant  $k_\eta$  occurring in Theorem 2.4. We give conditions for the non-vanishing of it, based on the observation that  $\text{Tp}(\eta)$  restricted to the smooth part  $\eta_0$  of  $\eta$  is the equivariant normal Euler class of  $\eta_0$ . This property was crucial in all the Thom polynomial calculations using the method of restriction equations in [Rim01].

Let  $V = \mathcal{E}(n, m)$ ,  $V^\sharp = \mathcal{E}(n, m + 1)$ ,  $G = \mathcal{K}(n, m)$ ,  $H = U(1)$ ,  $j : V \rightarrow V^\sharp$ ,  $\eta_0 \subset \eta$ ,  $\eta_0^\sharp \subset \eta^\sharp$  (case **A** or **B**),  $\text{Tp}(\eta) = q(c_1, c_2, \dots)$ , and  $\text{Tp}(\eta^\sharp) = p(c_1^\sharp, c_2^\sharp, \dots)$  be defined as above; and for  $X \subset Y$  let  $\nu_X$  denote the normal bundle of  $X$ . Observe that through  $j$  we can identify the bundle  $\nu_{\eta_0}$  as a subbundle of  $\nu_{\eta_0^\sharp}|_{j(\eta_0)}$ , hence for the  $G \times U(1)$ -equivariant Euler classes we have

$$e\left(\nu_{\eta_0^\sharp}|_{j(\eta_0)}\right) = e(\nu_{\eta_0}) \cdot e(M),$$

where  $M$  is the complementary subbundle of rank  $l^\sharp - l$  (Replacing  $G$  with the subgroup  $U(n) \times U(m)$ —it doesn't change the equivariant cohomology—we can assume that  $M$  is  $G$ -equivariant). Let  $r : H_G^*(V) \rightarrow H_G^*(\eta_0)$ , and  $r^\sharp : H_{G \times U(1)}^*(V^\sharp) \rightarrow H_{G \times U(1)}^*(\eta_0^\sharp)$  be the restriction homomorphisms. Then—since the cohomology class of a submanifold restricted to the submanifold itself is the Euler class of the normal bundle of the submanifold—we obtain that

$$(1) \quad p(j^*r^\sharp(c_1^\sharp), j^*r^\sharp(c_2^\sharp), \dots) = q(r(c_1), r(c_2), \dots) \cdot e(M).$$

Using the identity

$$1 + c_1^\sharp t + c_2^\sharp t^2 + \dots = (1 + yt)(1 + c_1 t + c_2 t^2 + \dots)$$

and Corollary 2.5, the left hand side can be written as

$$y^{l^\sharp - l} \cdot p^{[l^\sharp - l]}(r(c_1), r(c_2), \dots) + \text{terms of lower } y\text{-power}.$$

According to Theorem 2.4 it is further equal to

$$y^{l^\sharp - l} \cdot k_\eta \cdot q(r(c_1), r(c_2), \dots) + \text{terms of lower } y\text{-power}.$$

Thus from equation (1) we obtain that if the normal Euler class  $e(\nu_{\eta_0}) = q(r(c_1), r(c_2), \dots)$  is not 0, then  $k_\eta$  is the coefficient of  $y^{l^\sharp - l}$  in  $e(M)$ . Restricting the action of  $G \times U(1)$  to  $U(1)$  we obtain the following equivalent statement.

**Theorem 3.1.** *Suppose  $e(\nu_{\eta_0}) \neq 0$ . Then the  $U(1)$ -equivariant Euler class  $e_{U(1)}(M) = k_\eta \cdot y^{l^\sharp - l} \in H^*(BU(1)) = \mathbf{Q}[y]$ .*

The points of  $\eta_0$  are  $U(1)$ -invariant, hence the coefficient  $k_\eta$  is the product of the weights of the  $U(1)$ -action on a fiber of  $M$ . When  $\eta_0$  is the contact singularity corresponding to the local algebra  $Q$  the fiber of  $M$  is identified with the maximal ideal of the local algebra  $Q$  with scalar  $U(1)$ -action (see [Rim01]), that is, in this case  $e_{U(1)}(M) = 1$ .

It also follows from results in [Rim01] that the vanishing of  $e(\nu_{\eta_0})$  only depends on the local algebra of  $\eta_0$ , not on the particular dimensions  $n$  and  $m$ . Also, the Euler class  $e(\nu_{\eta_0})$  is not zero for any of the *simple* singularities considered in that paper—e.g. singularity types  $A_i$ ,  $I_{a,b}$ ,  $III_{a,b}$ , and more. (In fact for all singularities with known Thom polynomial the Euler class  $e(\nu_{\eta_0})$  is not zero.) Hence for all these singularities we proved that Theorem 2.4 holds with constant  $k_\eta = 1$ .

## 4. THOM SERIES

Now we use the  $d_i = c_{i+(m-n+1)}$  substitution to write the Thom polynomials of natural infinite sequences of singularities in a concise form.

**4.1. Contact singularities.** Let  $Q$  be a local algebra of a singularity. We will need three integer invariants of  $Q$  as follows: (i)  $\delta = \delta(Q)$  is the complex dimension of  $Q$ , (ii) the *defect*  $d = d(Q)$  of  $Q$  is defined to be the minimal value of  $b - a$  if  $Q$  can be presented with  $a$  generators and  $b$  relations; (iii) the definition of the third invariant  $\gamma(Q)$  is more subtle, see [Mat71, §6]. The existence of a *stable* singularity  $(\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^m, 0)$  with local algebra  $Q$  is equivalent to the conditions  $m - n \geq d$ ,  $(m - n)(\delta - 1) + \gamma \leq n$ . Under these conditions the codimension of the  $\mathcal{K}(n, m)$  orbit of a germ  $\eta : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^m, 0)$  with local algebra  $Q$  is  $(m - n)(\delta - 1) + \gamma$ .

Now we can apply Theorem 2.4 and Corollary 2.5 to the series of contact singularities with local algebra  $Q$ . Observe that Theorem 3.1 and the discussion after computes the value  $k_\eta = 1$ . Hence we obtain the following theorem.

**Theorem 4.1.** *Let  $Q$  be a local algebra of singularities defining Thom polynomials. Assume that the normal Euler classes of these singularities are not 0. Then associated with  $Q$  there is a formal power series (Thom series)  $\text{Ts}_Q$  in the variables  $\{d_i | i \in \mathbf{Z}\}$ , of degree  $\gamma(Q) - \delta(Q) + 1$ , such that all of its terms have  $\delta(Q) - 1$  factors, and the Thom polynomial of  $\eta : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^m, 0)$  with local algebra  $Q$  is obtained by the substitution  $d_i = c_{i+(m-n+1)}$ .  $\square$*

Even though there are powerful methods by now to compute individual Thom polynomials (i.e. finite initial sums of the Ts), finding closed formulas for these Thom series remains a subtle problem. Here are some examples.

**A<sub>0</sub>:**  $Q = \mathbf{C}$  (embedding). Here  $\delta = 1$ ,  $\gamma = 0$ , and

$$\text{Ts} = 1.$$

**A<sub>1</sub>:**  $Q = \mathbf{C}[x]/(x^2)$  (e.g. *fold*, *Whitney umbrella*). Here  $\delta = 2$ ,  $\gamma = 1$ , and

$$\text{Ts} = d_0.$$

**A<sub>2</sub>:**  $Q = \mathbf{C}[x]/(x^3)$  (e.g. *cuspidal cusp*). Here  $\delta = 3$ ,  $\gamma = 2$ , and [Ron72]

$$\text{Ts} = d_0^2 + d_{-1}d_1 + 2d_{-2}d_2 + 4d_{-3}d_3 + 8d_{-4}d_4 + \dots$$

**A<sub>3</sub>:**  $Q = \mathbf{C}[x]/(x^4)$ . Here  $\delta = 4$ ,  $\gamma = 3$ , and [BFR03, Thm.4.2]

$$\text{Ts} = \sum_{i=0}^{\infty} 2^i d_{-i} d_0 d_i + \frac{1}{3} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 2^i 3^j d_{-i} d_{-j} d_{i+j} + \frac{1}{2} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{i,j} d_{-i-j} d_i d_j,$$

where  $a_{i,j}$  is defined by the formal power series

$$\sum_{i,j} a_{i,j} u^i v^j = \frac{u \frac{1-u}{1-3u} + v \frac{1-v}{1-3v}}{1-u-v}.$$

**I<sub>2,2</sub>:**  $Q = \mathbf{C}[x, y]/(xy, x^2 + y^2)$ . Here  $\delta = 4$ ,  $\gamma = 4$ , and

$$\text{Ts} = \sum_{i=1}^{\infty} 2^{i-2} d_{-i} d_1 d_i - \sum_{i=1}^{\infty} 2^{i-1} d_{-i} d_0 d_{i+1} + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \binom{i+j-2}{i-1} d_{-i-j+1} d_i d_j.$$

Strictly speaking we have not proved the Thom series of  $I_{2,2}$ , just obtained overwhelming computer evidence for it. For an accurate proof (using the method of restriction equations from [FR04]) we would need to manipulate non-trivial resultant identities similar to those in [BFR03].

Recently P. Pragacz in [Pra05] used Schur functions and other powerful tools of symmetric functions in order to calculate the Thom polynomials of certain contact class singularities  $\eta : \mathbf{C}^n \rightarrow \mathbf{C}^m$  for any  $n$  and  $m$ . In particular he calculates the Thom polynomials of the singularities of the list above in terms of Schur functions. We do not see how to prove directly that the two results agree. We were informed by A. Szenes and G. Bérczi that they calculated Thom series for  $A_i$  for some values of  $i > 3$ .

**Remark 4.2.** A remarkable property of all the computed Thom series is that they have positive coefficients when written in the basis of Schur polynomials instead of the basis of Chern monomials. This was recently proved in [PW06].

**4.2. Thom-Boardman classes.** The Thom-Boardman singularity  $\Sigma^I$  is the closure of a contact singularity if the relative dimension  $m - n$  is large. In this case the constant  $k_\eta$  is 1, due to the results of Section 4.1. We conjecture that the constant  $k_\eta$  is 1 for Thom-Boardman singularities with arbitrary relative dimension  $m - n$ , too. However it is only known for the cases  $\Sigma^i$  and  $\Sigma^{i,j}$ .

**Theorem 4.3.** *For every  $r$  there is a formal power series  $\text{Ts}_{\Sigma^r}$  in the variables  $\{d_i | i \in \mathbf{Z}\}$ , of degree  $r(r-1)$ , such that all of its terms have  $r$  factors, and the Thom polynomial of  $\Sigma^r(n, m)$  is obtained by the substitution  $d_i = c_{i+(m-n+1)}$ .*

This is the statement we obtain by applying Theorem 2.4 to the corank  $r$  singularities (case **B** above with  $I = (r)$ ). It is not new though, since all these polynomials are known explicitly (Giambelli-Thom-Porteous):

$$\text{Tp}(\Sigma^r) = \det(d_{r-1+j-i})_{i,j=1,\dots,r},$$

that is, the Thom series in this case is finite. Also observe that to prove Theorem 4.3 we did not need the infinite dimensional spaces  $\mathcal{E}(n, m)$ , we could have started with  $\text{Hom}(\mathbf{C}^n, \mathbf{C}^m)$ .

The codimension of the set of germs  $\mathbf{C}^n \rightarrow \mathbf{C}^m$  with Thom-Boardman symbol  $(i, j)$  is  $(i + m - n)i + j((i + m - n)(2i - j + 1) - 2(i - j))/2$ . There are algorithms to calculate the Thom polynomials of second order Thom-Boardman singularities see [Ron72], [Kaz04], [KF05]. From [KF05] it also follows that  $k_\eta = 1$  for  $\Sigma^{i,j}$  singularities. Hence we have the following result about the structure of Thom polynomials of singularities of Thom-Boardman symbol  $(i, j)$ .

**Theorem 4.4.** *For every  $i \geq j$  there is a formal power series  $\text{Ts}_{\Sigma^{i,j}}$  in the variables  $\{d_i | i \in \mathbf{Z}\}$  of degree  $i(i-1) + j(2i^2 - ij - 3i + 3j - 1)/2$ , such that all of its terms have  $i(j+1) - \binom{j}{2}$  factors, and the Thom polynomial of  $\Sigma^{i,j}(n, m)$  is obtained by the substitution  $d_i = c_{i+(m-n+1)}$ .*

In general closed formulas for the  $\Sigma^{i,j}$  Thom series are not known, with the following exceptions. The Thom series of  $\Sigma^{i,1}$  is explicitly computed in Theorem 4.8 of [KF05]. A closed formula for the ‘initial term’ of the Thom series of  $\Sigma^{i,j}$  is calculated in 4.6 of [KF05]. Here ‘initial term’ refers to the terms not containing a  $d_u$  factor with  $u \leq -i + 1$ .

## 5. OTHER STABILIZATIONS: PRODUCT SINGULARITIES

Another way of looking at the results in Section 2 is that we related the Thom polynomial of a singularity  $\eta : (\mathbf{C}^n, 0) \rightarrow (\mathbf{C}^m, 0)$  to the Thom polynomial of  $\eta \times \xi$ , where  $\xi$  is the map

$(\mathbf{C}^0, 0) \rightarrow (\mathbf{C}^1, 0)$ . We can, however, choose other  $\xi$  maps (with some care), and relate the Thom polynomial of  $\eta$  with the Thom polynomial of  $\eta \times \xi$ .

Let us fix a germ  $\xi : (\mathbf{C}^a, 0) \rightarrow (\mathbf{C}^b, 0)$  whose right-left symmetry group is  $H$  (or  $H$  is a subgroup of the right-left symmetry group), with representations  $\lambda_0, \lambda_1$  on the source and the target spaces respectively. Let  $c(\xi)$  be the formal quotient Chern class  $c(\lambda_1)/c(\lambda_0) \in H^*(BH)$ . Consider the map  $j : \mathcal{E}(n, m) \rightarrow \mathcal{E}(n+a, m+b)$ ,  $f \mapsto f \times \xi$ . If  $\eta \subset \mathcal{E}(n, m)$  is the closure of a  $\mathcal{K}(n, m)$  orbit of codimension  $l$  and

- $\eta^\sharp := \overline{\mathcal{K}(n+a, p+b) \cdot j(\eta)}$  defines a Thom polynomial of degree  $l^\sharp$ , and
- $j^{-1}(\eta^\sharp) = \eta$ ,

then Theorem 2.1 gives the following relation between the Thom polynomials of  $\eta$  and  $j(\eta)$ .

**Theorem 5.1.** *Let the classes  $c_i^\sharp$  be defined by  $1 + c_1^\sharp + c_2^\sharp + \dots := (1 + c_1 + c_2 + \dots) \cdot c(\xi)$ . Suppose that substituting these classes into  $\text{Tp}(\eta^\sharp)$  results in  $\sum x_i \cdot y_i$ . That is, let*

$$\text{Tp}(\eta^\sharp) \left( (1 + c_1 + c_2 + \dots) \cdot c(\xi) \right) = \sum x_i \cdot y_i \quad \in \mathbf{Q}[c_1, c_2, \dots] \langle y_i \rangle,$$

where  $x_i \in \mathbf{Q}[c_1, c_2, \dots]$ , and  $y_i$  is an additive basis of  $H^*(BH)$ . Then  $x_i = 0$  for  $\deg x_i < l$ , and  $x_i = \text{constant} \cdot \text{Tp}(\eta)$  if  $\deg x_i = l$ .

The case studied in Section 2 is recovered as  $\xi : (\mathbf{C}^0, 0) \rightarrow (\mathbf{C}^1, 0)$ ,  $H = U(1)$ ,  $\lambda_1$  = the standard representation,  $\lambda_0$  = the 0-dimensional representation,  $c(\xi) = 1 + y \in H^*(BH)$ .

The stability of Thom polynomials of contact singularities under trivial unfolding corresponds to the following case:  $\xi : (\mathbf{C}^1, 0) \rightarrow (\mathbf{C}^1, 0)$ ,  $x \mapsto x$ ,  $H = U(1)$ ,  $\lambda_0 = \lambda_1$  = the standard representation.

Unfortunately, it is hard to find other  $\xi$ 's, for which one can easily check the two conditions above. If  $\xi$  is a complicated germ, then the set  $\eta^\sharp$  is usually within the realm of moduli of singularities, where the question of *what* defines a Thom polynomial is very hard (it reduces to the computation of Kazarian's spectral sequence [Kaz97], [FR04, Sect.10]). Hence we show an example with a simple  $\xi$ .

**Example 5.2.** Let  $\xi : (\mathbf{C}, 0) \rightarrow (\mathbf{C}, 0)$ ,  $x \mapsto x^2$ ; and let  $H = U(1)$  act by  $\lambda_0 = \rho$ ,  $\lambda_1 = \rho^{\otimes 2}$ , where  $\rho$  is the standard representation of  $U(1)$ . Let  $\eta \subset \mathcal{E}(n, n)$  be the closure of the 2-codimensional  $\mathcal{K}$ -orbit corresponding to the local algebra  $\mathbf{C}[x]/(x^3)$ , i.e. the orbit of  $(x_1, x_2, \dots, x_n) \mapsto (x_1^3, x_2, \dots, x_n)$ . Then  $\eta^\sharp = j(\eta)$  is the closure of the 7-codimensional  $\mathcal{K}(n+1, n+1)$  orbit corresponding to the local algebra  $\mathbf{C}[x, y]/(x^3, y^2)$ . The two conditions above hold and we obtain, that if

$$(2) \quad \text{Tp}(\eta^\sharp) \left( (1 + c_1 + c_2 + \dots)(1 + y - y^2 + y^3 - \dots) \right) = x_0 y^7 + x_1 y^6 + \dots + x_7 y^0,$$

then we must have  $x_0 = x_1 = 0$  and  $x_2 = \text{constant} \cdot \text{Tp}(\eta)$ . Indeed, these polynomials are known explicitly,  $\text{Tp}(\eta^\sharp) = 2(c_1^\sharp c_2^\sharp{}^3 - c_1^\sharp{}^2 c_2^\sharp c_3^\sharp + c_2^\sharp{}^2 c_3^\sharp + c_1^\sharp c_3^\sharp{}^2 - 2c_1^\sharp c_2^\sharp c_4^\sharp + c_3^\sharp c_4^\sharp - c_2^\sharp c_5^\sharp)$  [Por72], and thus the left hand side of (2) becomes  $2 \left( (c_1 + y)(c_2 + c_1 y - y^2)^3 - \dots \right) = 4(c_1^2 + c_2)y^5 + 2(c_1 c_2 - c_1^3 + 10c_3)y^4 + \dots$ . This is consistent with the fact that  $\text{Tp}(\eta) = c_1^2 + c_2$  (and the other  $y$  coefficients also belong to  $\mathcal{A}_\eta$ ).

Notice that Theorem 5.1 says that under the specified conditions we can calculate  $\text{Tp}(\eta)$  and  $\text{Tp}(\xi)$  if we know  $\text{Tp}(\eta \times \xi)$  but not in the other direction. Nevertheless it gives strong restrictions on the form of  $\text{Tp}(\eta \times \xi)$ .



The analogous ‘Thom-Sebastiani’ approach to finding relations between different Thom polynomials is more promising in finite dimensional settings, where we know the existence of Thom polynomials, and their properties are relevant in algebraic combinatorics. We plan to study the Thom polynomials for quiver representations from this perspective in the future.

**Remark 5.3.** There are other natural infinite series of pairs of Thom polynomials where Theorem 2.4 applies. For instance, we can consider the Thom polynomials of the orbits of  $S^2(\mathbf{C}^n)$  and  $S^2(\mathbf{C}^{n+1})$  with actions of  $GL(n)$  and  $GL(n) \times U(1)$ . Their Thom polynomial theory is worked out, see [JLP82], also [FR04, Sect. 5]. The constant  $k_\eta$  in this case, however, turns out to be different from 1.

## REFERENCES

- [AGLV98] V. I. Arnold, V. V. Goryunov, O. V. Lyashko, and V. A. Vasil’ev. *Singularity theory. I. Translated from the 1988 Russian original*. Encyclopaedia of Mathematical Sciences, Dynamical systems. VI. Springer, 1993, 1998.
- [BFR03] G. Bérczi, L. Fehér, and R. Rimányi. Expressions for resultants coming from the global theory of singularities. In L. McEwan, J.-P. Brasselet, C. Melles, G. Kennedy, and K. Lautier, editors, *Topics in Algebraic and Noncommutative Geometry*, number 324 in Contemporary Mathematics. AMS, 2003.
- [Boa67] J. M. Boardman. Singularities of differentiable maps. *Publ. math. de l’I.H.É.S.*, 33(2):21–57, 1967.
- [Dam72] J. Damon. Thom polynomials for contact class singularities. Ph.D. Thesis, Harvard, 1972.
- [FR04] L. M. Fehér and R. Rimányi. Calculation of Thom polynomials and other cohomological obstructions for group actions. In *Real and Complex Singularities (Sao Carlos, 2002)*, number 354 in Contemp. Math., pages 69–93. Amer. Math. Soc., 2004.
- [JLP82] T. Józefiak, A. Lascoux, and P. Pragacz. Classes of determinantal varieties associated with symmetric and skew-symmetric matrices. *Math. ISSR Izvestija*, 18:575–586, 1982.
- [Kaz97] M. É. Kazarian. Characteristic classes of singularity theory. In *The Arnold-Gelfand mathematical seminars*, pages 325–340. Birkhäuser Boston, 1997.
- [Kaz04] M. É. Kazarian. Derived Porteous-Thom classes and Thom polynomials for  $\Sigma^{a,b}$ -singularities. <http://www.mi.ras.ru/~kazarian/Vnmath/Ronga.ps>, 2004.
- [KF05] B. Kömüves and L. M. Fehér. On second order Thom-Boardman singularities. to appear in *Fundamenta Mathematica*, math.AG/0501498, 2005.
- [Mat71] J. N. Mather. Stability of  $C^\infty$  mappings VI: the nice dimensions. In C. T. C. Wall, editor, *Proceedings of Liverpool Singularities I*, LNM 192. Springer, 1971.
- [Por72] I. Porteous. The second-order decomposition of  $\Sigma^2$ . *Topology*, 11:325–334, 1972.
- [Pra05] P. Pragacz. Thom polynomials and Schur functions I. math.AG/0509234, 2005.
- [PW06] P. Pragacz and A. Weber. Positivity of Schur function expansions of Thom polynomials. math.AG/-0605308, 2006.
- [Rim01] R. Rimányi. Thom polynomials, symmetries and incidences of singularities. *Inv. Math.*, 143:499–521, 2001.
- [Rim02] R. Rimányi. On right-left symmetries of stable singularities. *Math. Z.*, 242:347–366, 2002.
- [Ron72] F. Ronga. La calcul des classes duales aux singularités de Boardman d’ordre 2. *Comm. Math. Helv.*, 47:15–35, 1972.
- [Tho56] R. Thom. Les singularités des applications différentiables. *Ann. Inst. Fourier, Grenoble*, 6:43–87, 1955–1956.

DEPARTMENT OF ANALYSIS, EOTVOS UNIVERSITY BUDAPEST, HUNGARY

*E-mail address:* lfeher@renyi.hu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL, USA

*E-mail address:* rimanyi@email.unc.edu